Classroom Cognitive and Meta-Cognitive Strategies for Teachers

Research-Based Strategies for Problem-Solving in Mathematics K-12

Florida Department of Education,
Bureau of Exceptional Education and Student Services
2010

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According to Polya (1957):

"One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else....The teacher should encourage the students to imagine cases in which they could utilize again the procedure used, or apply the result obtained" (p. 15-16).

The Problem-Solving Process:

Students can learn to become better problem solvers. Polya's (1957) "How to Solve It" book presented four phases or areas of problem-solving, which have become the framework often recommended for teaching and assessing problem-solving skills. The four steps are:

1. understanding the problem,
2. devising a plan to solve the problem,
3. implementing the plan, and
4. reflecting on the problem.

The following problem-solving process chart illustrates several strategies to be used to facilitate work with problem-solving. This process should be seen as a dynamic, non-linear and flexible approach. Learning these and other problem-solving strategies will enable students to deal more effectively and successfully with most types of mathematical problems. However, many other strategies could be added. These problem-solving processes could be very useful in mathematics, science, social sciences and other subjects. Students should be encouraged to develop and discover their own problem-solving strategies and become adept at using them for problem-solving. This will help them with their confidence in tackling problem-solving tasks in any situation, and enhance their reasoning skills. As soon as the students develop and refine their own repertoire of problem-solving strategies, teachers can highlight or concentrate on a particular strategy, and discuss aspects and applications of the strategy. As necessary, the students should develop flexibility to choose from the variety of strategies they have learned. We have provided some examples later in this document.

Step 1: Understanding the Problem

1. Read/Reread (for understanding)	2. Paraphrase (your own words)	3. Visualize (mentally or drawing)	4. Work in pairs or small groups
5. Identify Goal or Unknown	6. Identify Required Information	7. Identify Extraneous Information	8. Detect Missing Information
9. Define/Translate Use a dictionary	10. Check Conditions and/ or Assumptions	11. Share Point of View with Others	12. Others as Needed

Step 2: Devising a Plan to Solve the Problem

1. Estimate (quantity, measure or magnitude)	2. Revise 1st Estimate, 2nd estimate & so on	3. Share/Discuss Strategies	4. Work in pairs or small groups
5. Explain why the plan might work	6. Each try a common strategy or a different one	7. Reflect on Possible Solution Processes	8. Others as Needed

Step 3: Implementing a Solution Plan

1. Experiment with Different Solution Plans	2. Allow for "Mistakes"/Errors	3. Show all of my work Including partial solutions	4. Work in pairs or small groups
5. Discuss with others Different Solution Plans	6. Keep track and save all results/data	7. Compare attempts to solve similar problems	8. Find solution Do not give up
9. Implement your own solution plan	10. Attempts could be as important as the solution	11. Check your Answer(s)/Solution(s)	12. Others as Needed

Step 4: Reflecting on the Problem: Looking Back

1. Reflect on plan after you have an answer	2. Reflect on plan while finding the answer	3. Check if all problem conditions were made	4. Make sure I can justify/explain my answer
5. Check if correct assumptions were made	6. Check that I answer the problem question	7. Check if answer is unique or there are others	8. Reflect for possible alternative strategies
9. Reflect about possible more efficient process	10. Look for ways to extend the problem	11. Reflect on similarity/ difference to other prob.	12. Others as Needed

Step 1

Understanding the Problem

- ✓Survey, Question, Read (SQR)
- ✓Frayer Vocabulary Model
- ✓Mnemonic Devices
- ✓Graphic Organizers
- ✓ Paraphrase
- ✓Visualize
- ✓Cooperate Learning Groups
- ✓ Analyze Information

The first step in the Polya model is to understand the problem. As simple as that sounds, this is often the most overlooked step in the problem-solving process. This may seem like an obvious step that doesn't need mentioning, but in order for a problem-solver to find a solution, they must first understand what they are being asked to find out.

Polya suggested that teachers should ask students questions similar to the ones listed below:

- ✓ *Do you understand all the words used in stating the problem?*
- ✓ *What are you asked to find or show?*
- ✓ *Can you restate the problem in your own words?*
- ✓ *Can you think of a picture or a diagram that might help you understand the problem?*
- ✓ *Is there enough information to enable you to find a solution?*

Teachers should decide which strategies to use based on students' answers to these questions. For example, if Jason understands the meaning of all of the words in the problem, he does not need a vocabulary strategy, but if he cannot restate the problem, teaching him to paraphrase would be beneficial.

The following pages contain examples of strategies that teachers can use to support students through the first step in problem-solving.

Survey, Question, Read (SQR)

What is SQR?

The SQR strategy involves expansion and discussion between teacher and students. These discussions often lead to a better student understanding of the problem. This strategy was developed to help students arrive at their own solutions through rich discussion.

How do I use SQR?

Survey

- ✓ Read the problem
- ✓ Paraphrase in your own words

Question

- ✓ Question the purpose of the problem
    - What is being asked?
    - What are you ultimately trying to determine?

Read

- ✓ Reread the question
- ✓ Determine the exact information you are looking for
- ✓ Eliminate unnecessary information
- ✓ Prepare to devise a plan for solving the problem

Survey	Read the problem rapidly, skimming to determine its nature.
Question	Decide what is being asked; in other words, ask, “what is the problem?”
Read	Read for details and interrelationships.

(Leu, D. J., & Kinzer, C. K., 1991)

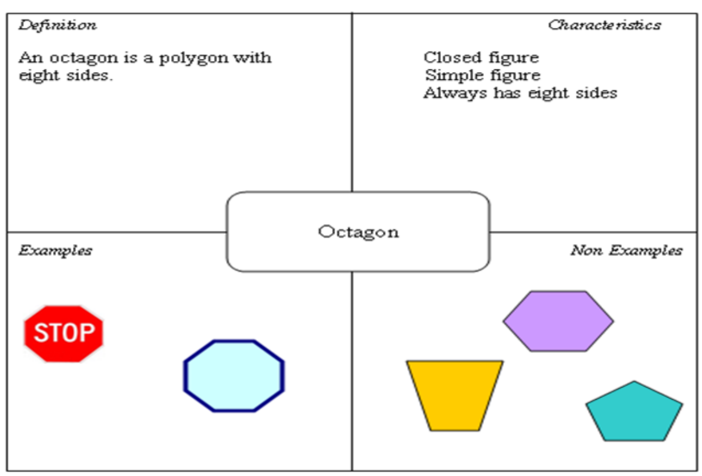
Frayer Vocabulary Model

What is the Frayer Vocabulary Model?

The Frayer model is a concept map which enables students to make relational connections with vocabulary words.

How do you use it?

1. Identify concept/vocabulary word.
2. Define the word in your own words.
3. List characteristics of the word.
4. List or draw pictures of examples and non-examples of the word.



(Frayer, Frederick, & Klauseier, 1969)

Mnemonic Devices

What are mnemonic devices?

Mnemonic devices are strategies that students and teachers can create to help students remember content. They are memory aids in which specific words are used to remember a concept or a list. The verbal information promotes recall of unfamiliar information and content (Nagel, Schumaker, & Deshler, 1986). Letter strategies include acronyms and acrostics (or sentence mnemonics). For example, "PEMDAS" is commonly used to help students remember the order of operations in mathematics.

How do you use letter strategy mnemonics?

1. Decide on the idea or ideas that the student needs to remember.
2. Show the student the mnemonic that you want them to use.
3. Explain what each letter stands for.
4. Give the students an opportunity to practice using the mnemonic.

Example 1: FIRST is a mnemonic device for creating mnemonics (Mercer & Mercer, 1998).

F - *Form a word* (from your concepts or ideas). Decide if you can create a word using the first letter of each word. Example: PEMDAS

I - *Insert extra letters to form a mnemonic* (only insert extra letters if you need them to create a word).

R - *Rearrange the first letters to form a mnemonic word.*

S - *Shape a sentence to form a mnemonic* (If you cannot form a word from the letters, use them to create a sentence). Example: Please Excuse My Dear Aunt Sally.

T -Try combinations of the first four steps to create a mnemonic.

Example 2: Ride is for problem-solving (Mercer & Mercer, 1993).

R - Read the problem correctly.

I - Identify the relevant information.

D - Determine the operation and unit for expressing the answer.

E - Enter the correct numbers and calculate.

Mnemonics are placed throughout this document to support students with different steps and strategies. Look for this symbol to mark each mnemonic. 

Graphic Organizers

What are graphic organizers?

Graphic organizers are diagrammatic illustrations designed to assist students in representing patterns, interpreting data, and analyzing information relevant to problem-solving (Lovitt, 1994, Ellis, & Sabornie, 1990).

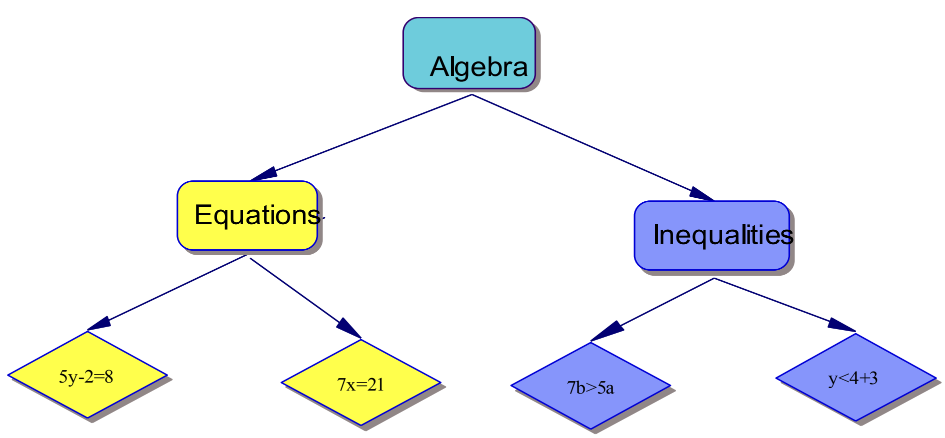
How do you use them?

1. Decide on the appropriate graphic organizer.
2. Model for the students using a familiar concept.
3. Allow the students to practice using the graphic organizer independently.

Examples:

Hierarchical Diagramming

These graphic organizers begin with a main topic or idea. All information related to the main idea is connected by branches, much like those found in a tree.



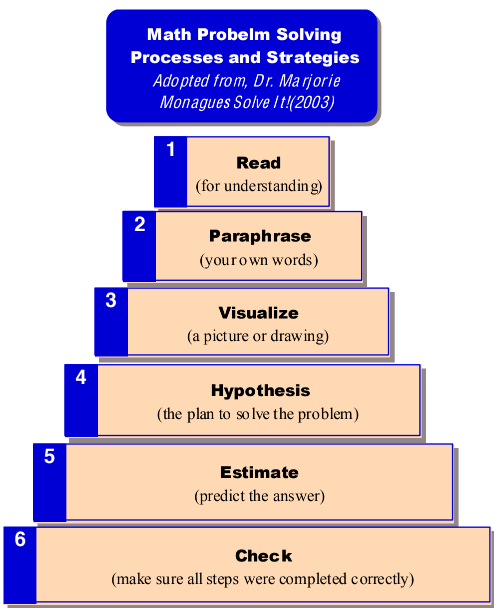
(This document can be found for classroom use in: Appendix A)

Sequence charts

These charts are designed to symbolize a sequence of procedures or events in a content area.

$$\begin{array}{rcl} 3x - 2 &=& 10 \\ \underline{\quad +2} && \underline{+2} \\ 3x &=& 12 \end{array} \qquad\qquad \frac{3x}{3} = \frac{12}{3} \qquad x = 4$$

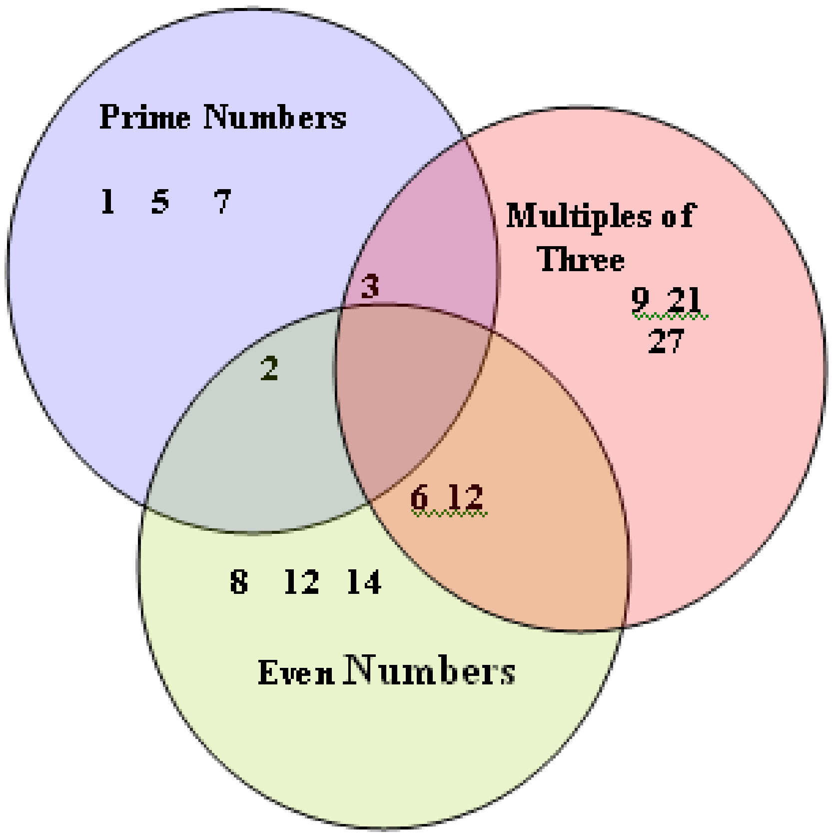
3x = 12
Divide by 3 → x = 4
Answer → 3(4) – 2 = 10
Check



(This document can be found for classroom use in: Appendix B)

Compare and contrast

These charts are designed to compare information across two or three groups or ideas.



(This document can be found for classroom use in: Appendix C)

Paraphrasing

What is the Paraphrasing Strategy?

The Paraphrasing Strategy is designed to help students restate the math problem in their own words, therefore strengthening their comprehension of the problem (Montague, 2005).

How do I use it?

1. Read the problem.
2. Underline or highlight key terms.
3. Restate the problem in your own words.
4. Write a numerical sentence.

Example #1:

Step 1 (Read the problem).

The middle school has 560 lockers available for the beginning of the school year. They have 729 students starting school. How many lockers will they be short on the first day of school?

Step 2 (Underline or highlight key terms).

The middle school has 560 lockers available for the beginning of the school year. They have 729 students starting school. How many lockers will they be short on the first day of school?

Step 3 (Restate the problem in your own words).

If there are 729 students and only 560 lockers, I need to know how much more 729 is than 560, therefore:

Step 4 (Write a numerical sentence).

729 - 560 = 169 lockers are still needed.

Example #2:

Step 1 (Read the problem).

A survey shows that 28% of 1,250 people surveyed prefer vanilla ice cream over chocolate or strawberry. How many of the people surveyed prefer vanilla ice cream?

Step 2 (Underline or highlight key terms).

A survey shows that 28% of 1,250 people surveyed prefer vanilla ice cream over chocolate or strawberry. How many of the people surveyed prefer vanilla ice cream?

Step 3 (Restate the problem in your own words).

If there are 1,250 students and 28% of them prefer vanilla ice cream, I need to know what 28% of 1,250 is. I also need to know that the word "of" means multiply. I can change 28% into a decimal or into a fraction. Therefore:

Step 4 (Write a numerical sentence).

1,250 x .28 = 350 people prefer vanilla ice cream.

OR 1,250 x 28 = 350 people prefer vanilla ice cream.

Visualization

What is visualization?

Visualization in mathematics is the practice of creating pictorial representations of mathematical problems. Students are asked to visualize and then draw the problem, allowing them to obtain a clearer understanding of what the problem is asking.

How do I teach visualization?

1. Read the problem.

2. Have the students underline important images in the problem.

3. Ask the students to draw a visual representation of the problem.

4. Write a numerical sentence.

** Be sure the students are drawing a representation of the problem, not just pictures of the items mentioned in the problem.

Example #1

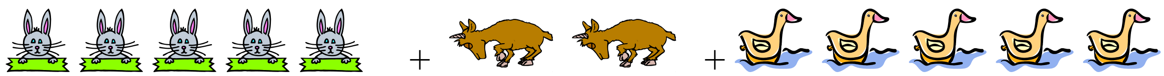
Step 1 (Read the problem).

There are 5 rabbits, 2 goats, and 6 ducks at the petting zoo. How many animals are at the petting zoo?

Step 2 (Have the students underline important images in the problem).

There are 5 rabbits, 2 goats, and 5 ducks at the petting zoo. How many animals are at the petting zoo?

Step 3 (Ask the students to draw a visual representation of the problem).



Step 4 (Write a numerical sentence).

5 + 2 + 5 = 12; *Answer: 12 animals are at the petting zoo.*

<u>Example #2</u>

Step 1 (Read the problem).

Dave was hiking on a trail that took him to an altitude that was 15 miles below sea level. Susan hiked to an altitude that was 8 miles above Dave. What was the final altitude for Susan's hike?

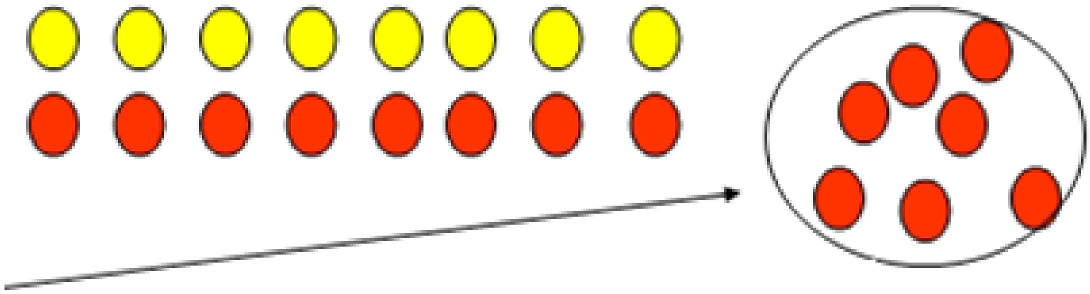
Step 2 (Have the students underline important images in the problem).

Dave was hiking on a trail that took him to an altitude that was <u>15 miles below</u> sea level. Susan hiked to an altitude that was <u>8 miles above</u> Dave. What was the final altitude for Susan's hike?

Step 3 (Ask the students to draw a visual representation of the problem).

Dave is 15 miles below sea level and Susan is 8 miles above Dave.

The final altitude for Susan can be represented by the expression $-15 + 8$.



Step 4 (Write a numerical sentence).

-7 was left over after taking out zero pairs, so the final altitude for Susan's hike is 7 miles below sea level.

Example #3

Karli has $12 to spend at the grocery store. She must buy 1 gallon of milk and some bags of snacks. The gallon of milk costs $4. How many bags of snacks can she buy if each bag costs $2?

Step 1 (Read the problem).

Karli has $12 to spend at the grocery store. She must buy 1 gallon of milk and some bags of snacks. The gallon of milk costs $4. How many bags of snacks can she buy if each bag costs $2?

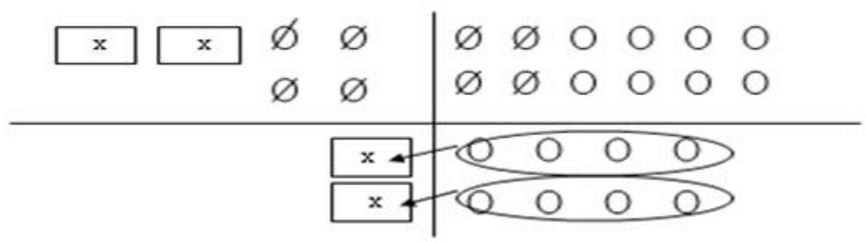
Step 2 (Have the students underline important images in the problem).

Karli has $12 to spend at the grocery store. She must buy 1 gallon of milk and some bags of snacks. The gallon of milk costs $4. How many bags of snacks can she buy if each bag costs $2?

Step 3 (Ask the students to draw a visual representation of the problem).

I know that the cost of the milk plus the cost of the snacks equals $12. The milk costs $4.

Each bag of snacks costs $2. I can write the equation 2x + 4 = 12 to represent the problem.



Step 4 (Write a numerical sentence).

There are four bags of snacks in each group, so x is equal to 4. Karli can buy 4 bags of snacks.

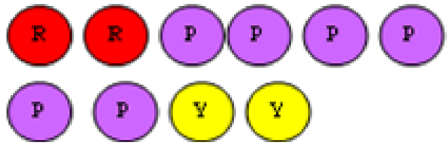
Example # 4

Step 1 (Read the problem).

Tarik has 2 red chips, 6 purple chips and 2 yellow chips in his bag. What fractional part of the bag of chips is red?

Step 2 (Have the students underline important images in the problem).

Tarik has 2 red chips, 6 purple chips and 2 yellow chips in his bag. What fractional part of the bag of chips is red?



Step 3 (Ask the students to draw a visual representation of the problem).

2 out of 10 chips are red.

Step 4 (Write a numerical sentence).

2/10 of the chips are red.

Cooperative Learning Groups

What is cooperative learning?

Students work in groups for a purpose assigned by the teacher. These activities allow students who differed in achievement, gender, race, and/or ethnicity to work together and learn from each other. Cooperative learning has demonstrated positive effects on student learning across numerous research studies (Johnson, Johnson, & Stanne, 2000).

There are FIVE critical elements for cooperative learning groups:

1. Positive interdependence
2. Individual accountability
3. Group processing
4. Social skills
5. Face-to-face interaction

How do you teach cooperative learning?

- ✓ Decide on the size and the members of each group (heterogeneous groups).
- ✓ Arrange the room for teacher monitoring and student face-to-face interactions without disruptions to other groups.
- ✓ Plan instructional materials to promote positive interdependence.
- ✓ Assign roles to ensure interdependence and accountability to each member.
- ✓ Explain the academic task.
- ✓ Structure and provide feedback towards positive social skills.
- ✓ Structure individual accountability.

Example # 1:

Activity: Roundtable

- ✓ Students sit in teams of 4.
- ✓ Each team of four gets 1 piece of paper and 1 pen.
- ✓ Teacher poses a problem.
- ✓ First student does 1st step/part of the problem, thinking aloud as they work.
- ✓ Student 1 passes paper to student 2, who checks the work and praises or re-teaches student 1.
- ✓ Student 2 does the next step of the problem, thinking aloud as they work.
- ✓ Process continues until a solution is reached.
- ✓ Good for problems with multiple steps or generating lists.

Evaluating Expressions

Person 1: evaluate (2x +y)/3
Person 2: evaluate 2x– y
Person 3: evaluate (0.5x)(y)
Person 4: evaluate x / (y/7)

X = 24

Y = 42

Analyze the Information

How do you teach your students to analyze and review the data?

Students work in pairs. Each group/pair should complete the meta-cognitive checklist below. After they have checked off each stage of step one, they should answer the problems asked below the checklist. After answering all four questions, they should be ready to move on to devising a plan to understand the problem.

Metacognitive Chart for Analyzing Data

Directions:
As you complete each phase of step1 in the problem solving process, check the appropriate box below.

1. Read/Reread	☐
2. Paraphrase (restate in your own words)	☐
3. Visualize (Draw a picture to represent the problem)	☐
4. Worked with a group to discuss the problem.	☐

After you have check off all four steps, answer the following questions:

1. What is the problem asking or what question am I trying to answer?
2. What information is still missing?
3. What type of mathematical computation will I need to use to solve the problem?
4. Does everyone in my group agree on these answers?

If everyone does not agree, be sure to write down all extra responses in the space below (you may need them for step 2).

(This document can be found for classroom use in: Appendix D)

Step 2

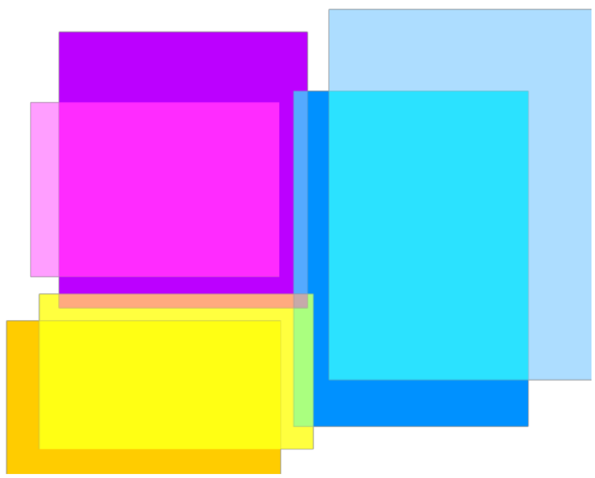
Devising a Plan to Solve the Problem

- ✓ Hypothesizing
- ✓ Estimating
- ✓ Discussing/Sharing Strategies
  - Guess and Check
  - Make an Organized List
  - Look for a Pattern
  - Eliminate Possibilities
  - Use Logical Reasoning
  - Draw a Picture
  - Use a Formula
  - Work Backwards
- ✓ Explaining the Plan

The second step in the Polya model requires that the learner devise a plan for solving the problem. They should find the connection between the data and the unknown. By the end of this step they should have a plan for the solution.

Questions the learner should consider are:

- ✓ Have you seen the same type of problem in a slightly different form?
- ✓ Try to think of a familiar problem having the same or a similar unknown.
- ✓ Could the problem be restated differently?
- ✓ Did you use all the data?



The following pages contain examples of strategies that teachers can use to support students through the second step in problem-solving: hypothesizing.

Hypothesizing

What is hypothesizing for problem-solving?

When hypothesizing, students are deciding on the best path for solving the problem. They are deciding on how to set up the problem and which operations are necessary (Montague, 2005).

1. read the problem,
2. paraphrase the problem, and
3. list the most appropriate methods for solving the problem.

How do you teach hypothesizing?

1. Teachers should model hypothesizing for the students.
2. Students should then work on creating hypothesis in a group.
3. Allow students to practice independently (quizzes, verbal responses, etc.).

- ✓ Explain criteria for success
- ✓ Monitor and reinforce appropriate and specified behaviors
- ✓ Provide task assistance, as needed, to the entire group
- ✓ Intervene to teach collaborative skills
- ✓ Provide closure to the lesson

Estimating

What is estimating for problem-solving?

When estimating for problem-solving, students try out the calculations from their hypothesis by rounding up or down to determine a possible solution to the problem. Some students calculate in their head while others use paper and pencil (Montague, 2005).

How do you teach your students estimating for problem-solving?

Because some students estimate in their head, while others estimate on paper, it is probably a good idea to have your students use paper and pencil at first. This way, you will know who needs more instruction. Have each member of the groups complete the worksheet for estimating. After completing the worksheet, ask the students to compare answers with other members of their group. After comparing answers, they should write down their final estimate at the bottom of the page.

Estimation Workspace for Problem Solving

Directions:

1. Write your hypothesis for the problem (you can write two if you have not decided on a final hypothesis).
2. In the workspace below your hypothesis, write your estimated answer. Check to see if it makes sense in the problem, and adjust it up or down.

Hypothesis #1	
Possible Solution #1	Possible Solution #2

Hypothesis #1	
Possible Solution #1	Possible Solution #2

Compare your answers with your group members and write down your final estimate below. ⬇

Final Estimate

(This document can be found for classroom use in: Appendix E)

Discuss/Share Strategies

According Polya (1957) there are many logical ways to solve problems. The skill at choosing an appropriate strategy is best learned by solving many problems. Students will find choosing a strategy increasingly easier with practice. Some possible strategies might include:

- ✓ guess and check
- ✓ make an organized list
- ✓ eliminate possibilities
- ✓ use symmetry
- ✓ consider special cases
- ✓ use direct reasoning
- ✓ solve an equation
- ✓ look for a pattern
- ✓ draw a picture
- ✓ solve a simpler problem
- ✓ use a model
- ✓ work backward
- ✓ use a formula
- ✓ be ingenious
- ✓ consider extremes

Guess and Check

What Is It?

"Guess and Check" is a problem-solving strategy that students can use to solve mathematical problems by guessing the answer and then checking that the guess fits the conditions of the problem.

How do you teach guess and check?

Step 1

Introduce a problem to students that will require them to make and then check their guess to solve the problem. For example, the problem:

> *Ben knows 100 baseball players by name. Ten are Yankee; the rest are Marlins and Braves. He knows the names of twice as many Marlins as Braves. How many Marlins does he know by name?*

Step 2

The students should find the key pieces of information needed to find the answer. This may require reading the problem more than once, and/or students paraphrasing the problem.

> *For example, "I know there are twice as many Marlins as Braves. There are 10 Yankees. The number of Marlins and Braves should equal 90."*

Step 3

Have students use a table or chart to implement the “guess and check strategy”.

Guess Number	Marlins	Braves	Yankees	Total
First Guess	10	20	10	40

Guess a greater number of Blue Jays.

Guess Number	Marlins	Braves	Yankees	Total
First Guess	10	20	10	40
Second Guess	20	40	10	70

Now guess a greater number of Blue Jays.

Guess Number	Marlins	Braves	Yankees	Total
First Guess	10	20	10	40
Second Guess	20	40	10	70
Third Guess	40	80	10	130

Now guess a number lesser than 40 and greater than 20.

Guess Number	Marlins	Braves	Yankees	Total
First Guess	10	20	10	40
Second Guess	20	40	10	70
Third Guess	40	80	10	130
Fourth Guess	30	60	10	100

That is the answer.

Step 4

Check.

Read the problem again to be sure the question was answered.

Yes, I found the number of Blue Jays.

Check the math to be sure it is correct.

30 doubled equals 60. 30 + 60 + 10 = 100.

Make an Organized List

What is an organized list?

An organized list is a problem-solving strategy which enables students to organize the data and visually consider their options when answering a problem. “In the attempt to produce an organized list, students will encounter frequent and repeated patterns.” (Muckerheide et al., 1999).

How do you teach students to make an organized list?

*Step 1 (*Read the problem).

The community of gnomes in the Magic Forest is upset because their forest is being bulldozed for a shopping mall. The little people are moving far away, too far to walk. They are going in the boats they made from the bark of trees. Each boat can hold up to and including 100 grams and stay afloat. The gnomes come in five different weights: 60 grams (senior citizens), 40 grams (adults), 20 grams (teenagers), 10 grams (children), and 5 grams (babies). What are the possible combinations of gnomes that can safely be put into a boat?

*Step 2 (*Restate the question in your own words).

What are all of the combinations of gnomes that can go in the boat as long as one is a 60 pound gnome?

*Step 3 (*Determine important information).

Weights of the gnomes, weight that the boat can hold.

*Step 4 (*Create an organized list).

If you begin with one 60-pound gnome, how many different combinations could you make to get to the 100 pounds?

*Step 5 (*Implement a solution).

(combinations for 1 60 lb gnome)

<u>60</u>	<u>40</u>	<u>20</u>	<u>10</u>	<u>5</u>
1	1	0	0	0
1	0	2	0	0
1	0	1	2	0
1	0	1	1	2
1	0	1	0	4
1	0	0	4	0
1	0	0	3	2
1	0	0	2	4
1	0	0	1	6
1	0	0	0	8

<u>Example 2</u>:

Lena is helping her father empty the coins from the commercial washing machines he services. Lena's father lets her count the coins in the machine with the smallest amount of money. The washing machines take only 50-cent pieces, quarters, and dimes. If Lena counts <u>$2.00 in coins</u>, how many different combinations of coins could Lena have counted? Find out the possible combinations of 50 cent pieces, quarters, and dimes that add up to two dollars.

50-cent pieces	quarters	dimes	Total Cost ($)
4	0	0	2.00
3	2	0	2.00
3	1	2	1.95
2	2	5	2.00
2	1	7	1.95
2	0	10	2.00
1	4	5	2.00
1	3	7	1.95
1	2	10	2.00
0	8	0	2.00
0	6	5	2.00
0	4	10	2.00
0	2	15	2.00
0	0	20	2.00

11 combinations are possible.

* Students should share solutions with the class in order to be sure that no combinations have been left out.

Look for a Pattern

What is a pattern?

Some problems can be solved by recognizing a pattern. Making a table can help the students recognize the pattern.

How do you teach students to look for a pattern?

*Step 1 (*Read the problem).

Daniel arranged loaves of bread on 6 shelves in the bakery. He put 1 loaf on the top shelf, 3 loaves on the second shelf, and 5 loaves on the third shelf. If he continues this pattern, how many loaves did Daniel put on the 6th shelf?

*Step 2 (*Make a table and look for a pattern).

Shelf	1	2	3	4	5	6
Loaves	1	3	5	7	9	11

*Step 3 (*Look for the pattern).

There are 2 more loaves on each shelf. The table is completed to shelf 6.

*Step 4 (*Solve the problem).

Daniel put 11 loaves on shelf 6.

Example 2:

A man was very overweight and his doctor told him to lose 36 pounds. If he loses 11 lbs the first week, 9 lbs the second week, and 7 lbs the third week, and he continues losing at this rate, how long will it take him to lose 36 lbs?

Make a table and look for a pattern.

Week	Pounds Lost
1	11
2	9 (20 lbs)
3	7 (27 lbs)
4	5 (32 lbs)
6	3 (35 lbs)
7	1 (36 lbs)

Each week, the man loses 2 pounds less than he did the previous week. Extend the pattern out to 7 weeks. It takes the man 7 weeks to lose 36 pounds.

Eliminating Possibilities

What is it?

Eliminating possibilities is a strategy where students use a process of elimination until they find the correct answer. This is a problem-solving strategy that can be used in basic math problems or to help solve logic problems.

How do you teach eliminating?

Step 1

Introduce a problem to students that will require them to eliminate possibilities in order to solve the problem. For example:

> *In the game of football, a team can score either a touchdown for six points or field goal for three points. If a team only scores touchdowns or field goals but does not get any extra points (no points for an extra kick) what scores cannot be achieved if the team scored under 30 points?*

Step 2

Let the students know that the first step is understanding the problem. They should identify the important pieces of information necessary for solving the problem. This may require paraphrasing.

Step 3

In this problem, students understand that there is a finite set of possible answers. Students will have to find all of the possible answers and then narrow down the list according to the criteria in the problem.

The score can be 1 through 29. The score must be a multiple of 3 or 6.

Step 4

The strategy of eliminating possibilities can be used in situations where there is a set of possible answers and a set of criteria the answer must meet.

- ✓ First, list the numbers 1 through 29, because the problem states that the score was less than 30.

  1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

- ✓ Next, eliminate answers that are not possible solutions. Work through each criteria to find the solution.
- ✓ Any multiple of six would be a possible score of the game. If the team only scored touchdowns, they could score 6, 12, 18, 24 and so on. Therefore, all multiples of six should be eliminated.

  1 2 3 4 5 * 7 8 9 10 11 * 13 14 15 16 17 * 19 20 21 22 23 * 25 26 27 28 29

- ✓ Any multiple of three would be a possible score of the game. If a team scored only field goals, they could score 3, 6, 9, and so on. Therefore, all multiples of three should be eliminated.

  1 2 * 4 5 * 7 8 * 10 11 * 13 14 * 16 17 * 19 20 * 22 23 * 25 26 * 28 29

- ✓ The answer to the problem is that the following scores could not be the score of the game:

  1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26, 28, 29

What is logical reasoning?

Logical reasoning is a problem-solving strategy that involves the use of Venn diagrams or charts to help students use logic to solve a problem.

How do you teach logical reasoning?

Step 1

Students should read the problem and paraphrase if necessary.

> *Intramural sports allows students to participate in several different sports. 32 children joined in the activities. 19 played soccer and 15 played basketball. How many children played both sports?*

Step 2

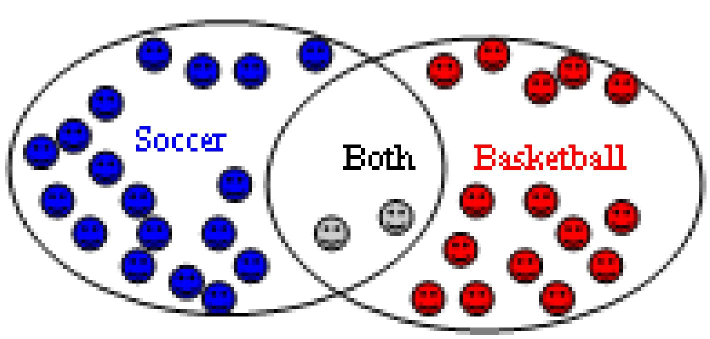
Students should ask themselves "what do I know about this problem?" Logic tells me that at least some children played both sports, while other played only one.

Step 3

A Venn diagram would be a logical way to sort this information.

Step 4

Draw the Venn diagram.



Put 19 counters inside the soccer circle and 15 counters inside the basketball circle.

With only 32 counters, 2 of them must be placed inside *both* circles.

The answer is 2 children played both sports.

Example 2

Three popular courses at a local high school are geography, art, and science. A review of the schedules of 200 students revealed that 70 have geography, 80 have science, 60 have art, 35 have geography and science, 33 have geography and art, 31 have science and art, and 15 have all three classes. How many of the 200 students have none of these classes?

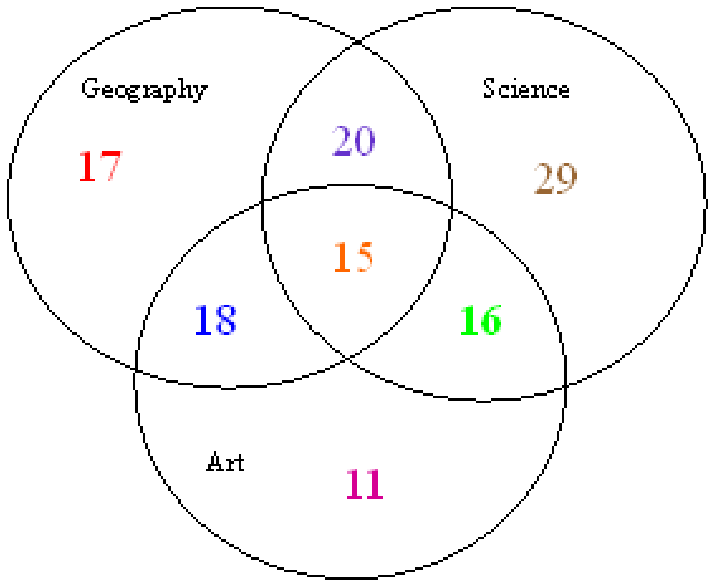
Start by putting 15 students in the center.

For geography and science, there are 20 students (35 total minus 15 in all three).

For geography and art, there are 18 students (33 total minus 15 in all three).

For science and art, there are 16 students (31 total minus 15 in all three).

Next, subtract to get the totals for geography (70- [15+20+18]), science (80- [15+20+16]), and art (60- [18+15+16]).



To find the final answer, subtract 200 - [17+20+18+15+29+16+11] = <u>74.</u>

Draw a Picture

What is "drawing a picture"?

Drawing a picture is a strategy that incorporates the use of drawing pictures to represent the problem. This strategy is frequently used to help solve some problems.

How do you **teach "drawing a picture" for** problem-solving?

Step 1

Students should read the problem carefully and paraphrase if necessary.

> *Four students were standing in line at the library. Javier was behind Timothy. Isaiah was between Daniel and Timothy. Daniel was in front of Colin. A book was on the floor near the student who was in the back of the line. Who was in the back of the line?*

Step 2

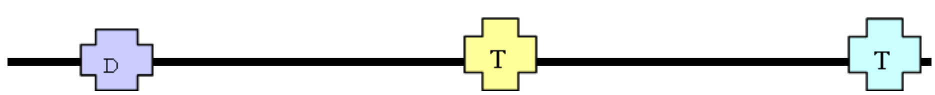
Students should draw a picture that represents the problem. For example, the students should draw a line.

Back  Front

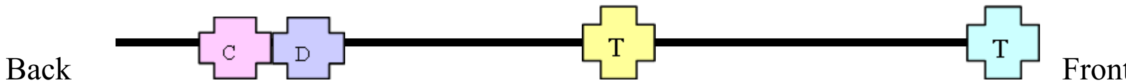
3. Next they should draw the representation of Daniel being in back of Timothy.

Back  Front

4. Now add Isaiah between Daniel and Timothy.

Back  Front

5. Last, add Colin behind Daniel.



Step 3

Answer the problem.

Colin is in the back of the line!

STAR

S -Search the word problem.

T -Translate the words into picture or equation.

A -Answer the problem.

R -Review the solution.

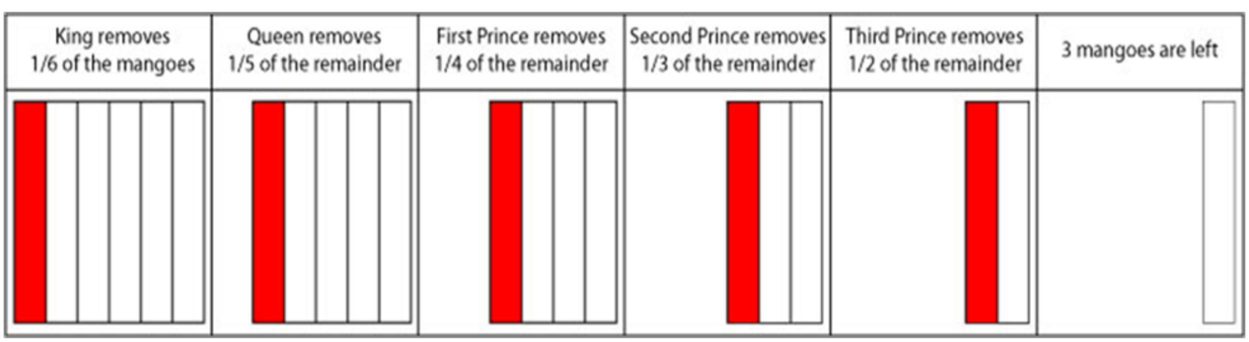
Example 2:

One night the King couldn't sleep, so he went down into the Royal kitchen, where he found a bowl full of mangoes. Being hungry, he took 1/6 of the mangoes. Later that same night, the Queen was hungry and couldn't sleep. She, too, found the mangoes and took 1/5 of what the King had left. Still later, the first Prince awoke, went to the kitchen, and ate 1/4 of the remaining mangoes. Even later, his brother, the second Prince, ate 1/3 of what was then left. Finally, the third Prince ate 1/2 of what was left, leaving only three mangoes for the servants.

How many mangoes were originally in the bowl?

Solution:

- ✓ Start by drawing a rectangle to represent all mangoes in the original pile prior to the removal of any of them.
- ✓ Since the King took one-sixth of this pile, divide the rectangle into six equal strips and "remove" one strip.
- ✓ Five strips remain, from which the Queen removed one-fifth, so this one-fifth is also represented by one of the original strips.
- ✓ Continuing, when the first Prince removes one-fourth of what is left, the one-fourth is represented by one of the strips.
- ✓ Similarly, the one-third, one-half, and 3 remaining mangoes are each represented by a strip.
- ✓ Since the 3 mangoes equal one strip and six strips were involved to start, the number of original mangoes must have been 6 × 3 = 18.



**Stonewater, Jerry (1994). Classic middle grades problems for the classroom. *Mathematics Teaching in the Middle School.*

Using a Formula

What is using a formula?

Using a formula is a problem-solving strategy that students can use to find answers to math problems. To solve these problems, students must choose the appropriate formula and substitute data in the correct places of a formula. Using a formula is a problem-solving strategy that can be used for problems that involve converting units or measuring geometric objects. Also, real-world problems such as tipping in a restaurant, finding the price of a sale item, and buying enough paint for a room all involve using formulas.

How do you teach students to use a formula?

1. Students should read the problem carefully and paraphrase if necessary.

   A rectangle has an area of 40 square meters. If the perimeter of the rectangle is 26 meters, what are the length and the width of the rectangle?

2. Here are a few formulas that students can use to solve this problem:

   A = L x W 40 = L x W

   2W + 2L = P 2W + 2L = 26

   (Note: L and W can be interchanged in this problem.)

3. Choose a formula.

4. Solve the problem.

   A = L x W 40 = L x W

   40 is a product of 2 and 20, 4 and 10, or 5 and 8.

   2W + 2L = P 2W + 2L = 26

The numbers 5 and 8 are the two numbers that work for both formulas.

Mnemonic for solving equations

<u>CAP</u>

C -Combine like terms.

A -Ask yourself, “How can I isolate the variable?”

P -Put the value of the variable in the initial equation and check if the equation is balanced.

Work Backwards

What is working backwards?

Working backwards is the strategy of undoing key elements in the problem in order to find a solution.

How do you teach students to work backward?

Step 1

Students should read the problem carefully and paraphrase if necessary.

> *The castle kitchen servants brought in 4 pies left over from the feast. 12 pies were eaten at the feast. Queen Mab took 2 home with her. How many pies did the servants bring into the feast at the beginning?*

Step 2

Account for all the pies that were eaten or taken home.

12 + 2 = 14

Step 3

Add the 4 pies that were left over.

14 + 4 + 18

Step 4

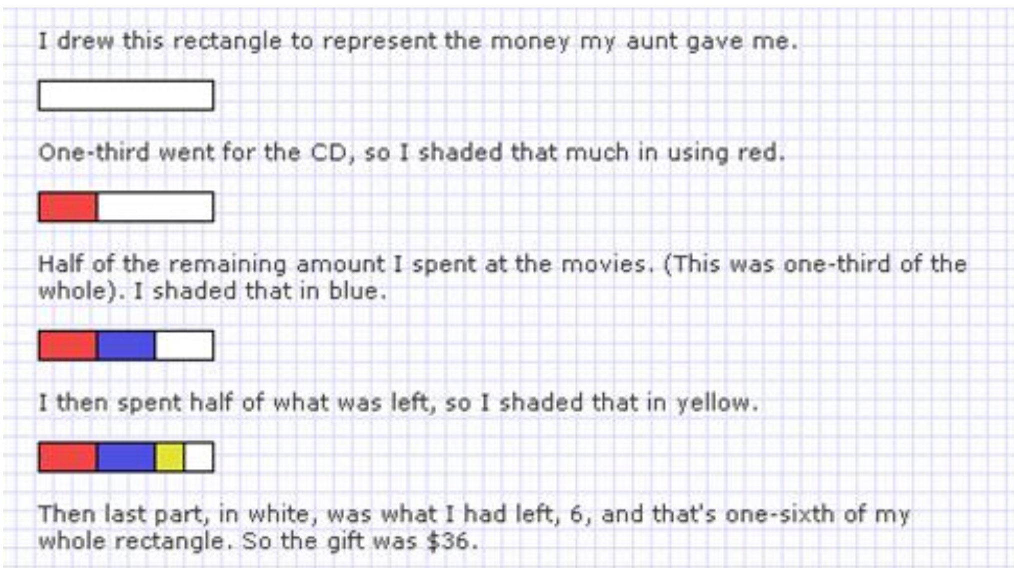
Solve the problem.

There must have been 18 pies at the start of the feast.

Example 2:

"My favorite aunt gave me some money for my birthday. I spent one-third of it on a new CD. I spent half the remainder to take my friend to the movies. Then I bought a magazine with half of what was left. When I went home, I still had $6. How much did my aunt give me for my birthday?"

One way to solve this problem combines drawing a diagram and working backward.



Explain the Plan

What is the explanation of the plan?

After the students have reviewed all of the strategies and decided on a plan, they should write down the plan. This is a good time to have the groups share with the class. Some groups may have decided on different methods for solving the problem, and the insight may be helpful to students who were still uncertain of the plan.

How do you have the students explain their plan?

While there are many ways to have the students share their plan, several ideas are listed below:

- ✓ Use chart paper and have the students write it down and hang it around the classroom.
- ✓ Allow the groups to come to the board and write/explain their plan.
- ✓ Ask each group to create a “handout” describing their plan. Use Microsoft Word or Publisher.
- ✓ Allow students to create a brief PowerPoint presentation explaining their plan.

Step 3

Implementing a Solution Plan

- ✓ Experiment with Different Solution Plans
- ✓ Allow for Mistakes/Errors
- ✓ Work Collaboratively
- ✓ Implement your own solution plan
- ✓ Check your Answer(s)/Solution(s)

Implement Your Own Solution Plan

After students have decided on a specific plan, they should follow the steps outlined below.

1. Solve the problem using your plan.
2. Be sure to double check each step.
3. If the plan is not working after a few attempts, try a different plan.
4. Allow for mistakes (remember the plan may need some revision).
5. Check your answer.

Allow students to use the following "Implementing a Solution Guide".

Implementing Your Solution

Step 1. Solve the problem using your plan.

Step 2. Be sure to double check each step.

Step 3. If the plan is not working after a few attempts, try a different plan.

Step 4. Allow for mistakes (remember the plan may need some revision).

Step 5. Check your answer. (Plug the answer back into the problem and decide if it makes sense).

(Appendix F)

Step 4

Reflecting on the Problem

✓Reflect on the plan

Reflect on the Plan

After students have completed their problem and come up with a solution they are satisfied with, they should reflect on the problem-solving process. Much can be gained by taking the time to reflect and look back at what you have done, what worked and what didn't. Doing this will enable you to predict what strategy to use to solve future problems (Polya, 1957).

Questions for reflection:

- ✓ Can you check the result?
- ✓ Can you check the argument?
- ✓ Does your answer make sense? (Can Uncle Fredrick really be –2 years old?)
- ✓ Did you answer all parts of the question?
- ✓ What methods worked? What methods failed? What did you learn from completing this problem?
- ✓ Could I have solved this problem another way?
- ✓ Was there an easier way to solve this problem?
- ✓ If I encountered a similar problem in the future, how could I better solve it?

Students should work within their collaborative group to answer and reflect on the answers to these questions. Students' reflections could be recorded in a math journal, possibly an electronic version, or completed on activity cards.

Reflect on your participation in the problem solving process:

My answer made sense because…

My method worked because…

I learned that I…

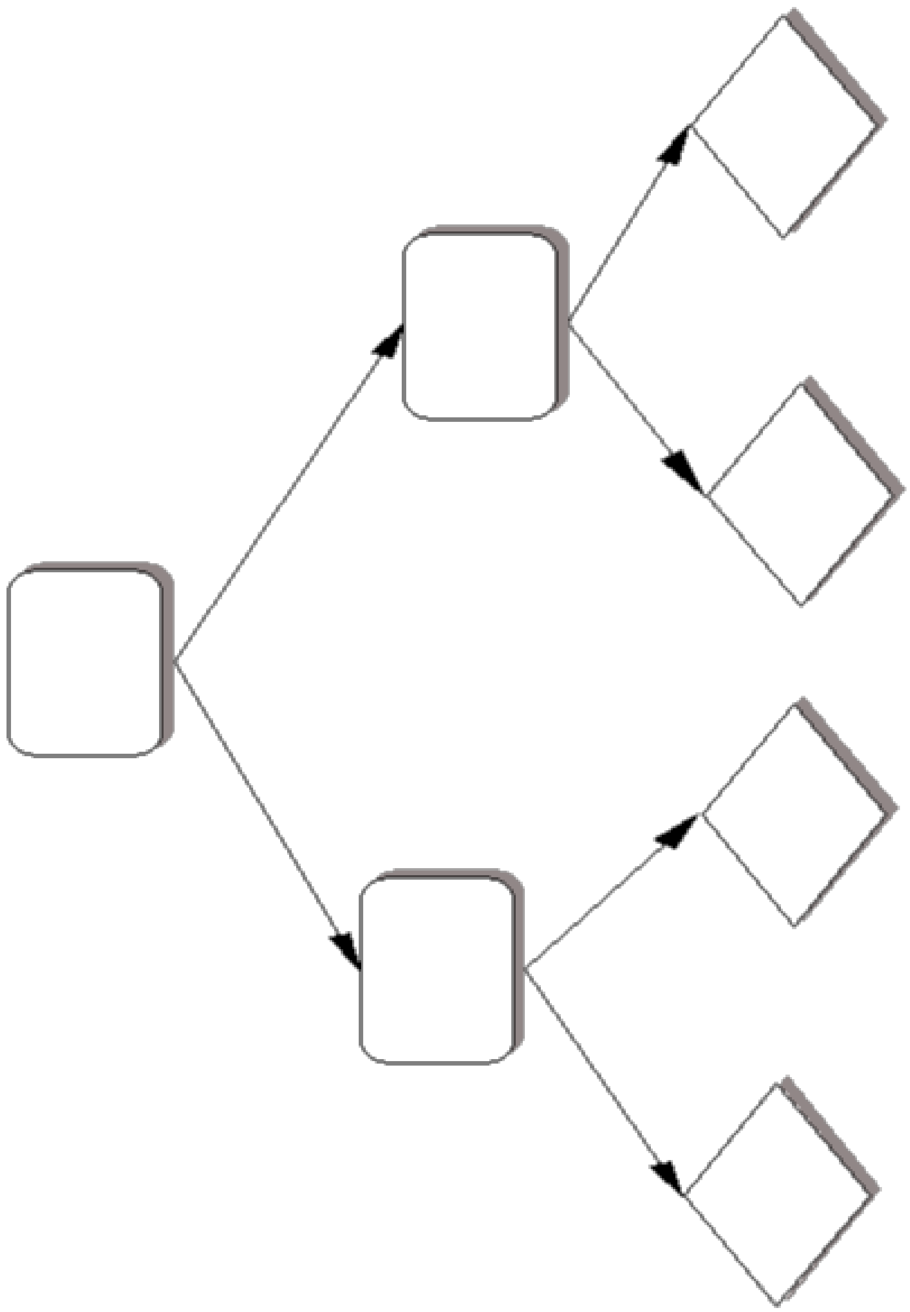
I was surprised that I…

(Appendix G)

Appendices

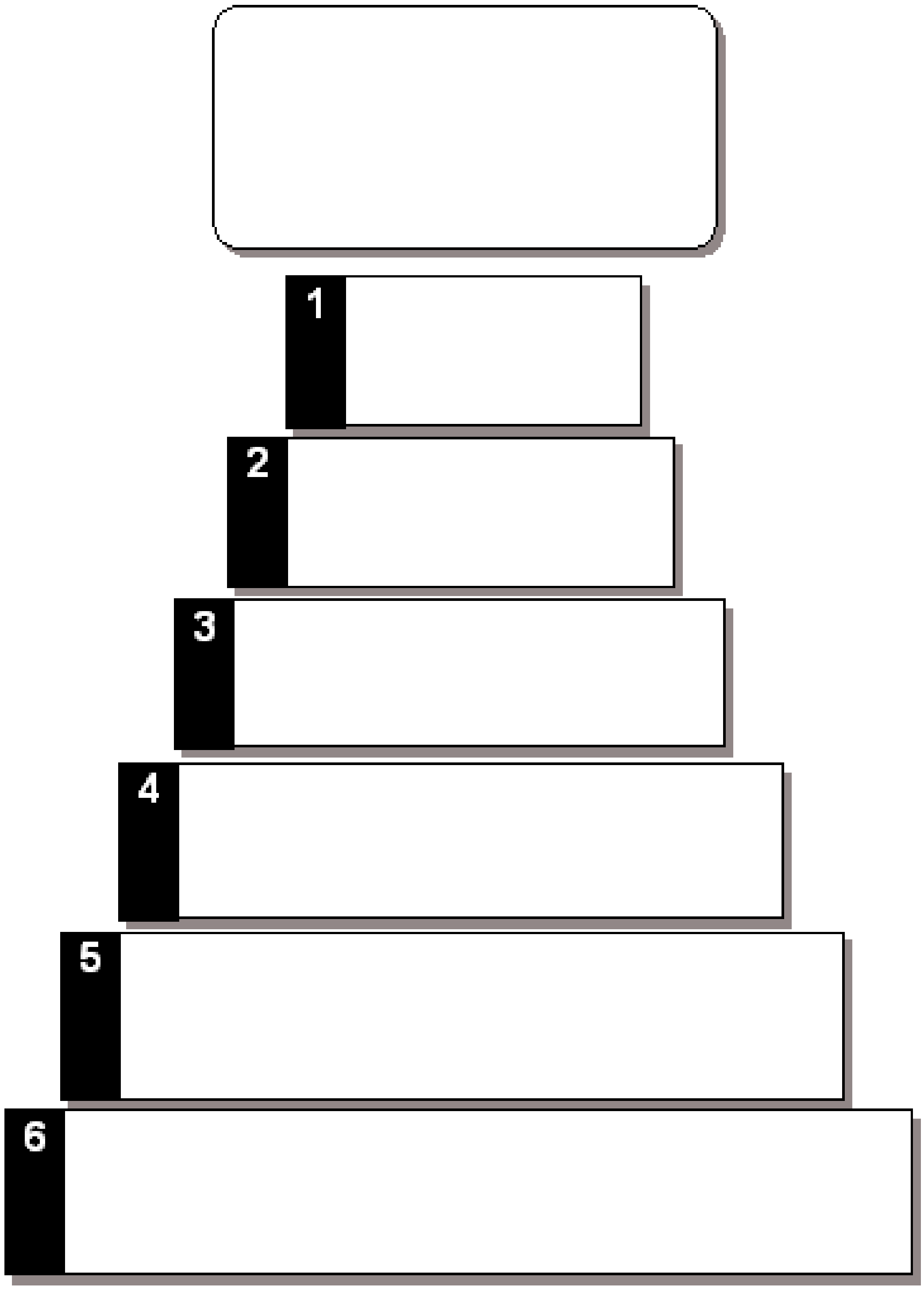
Appendix A

Hierarchical Diagram



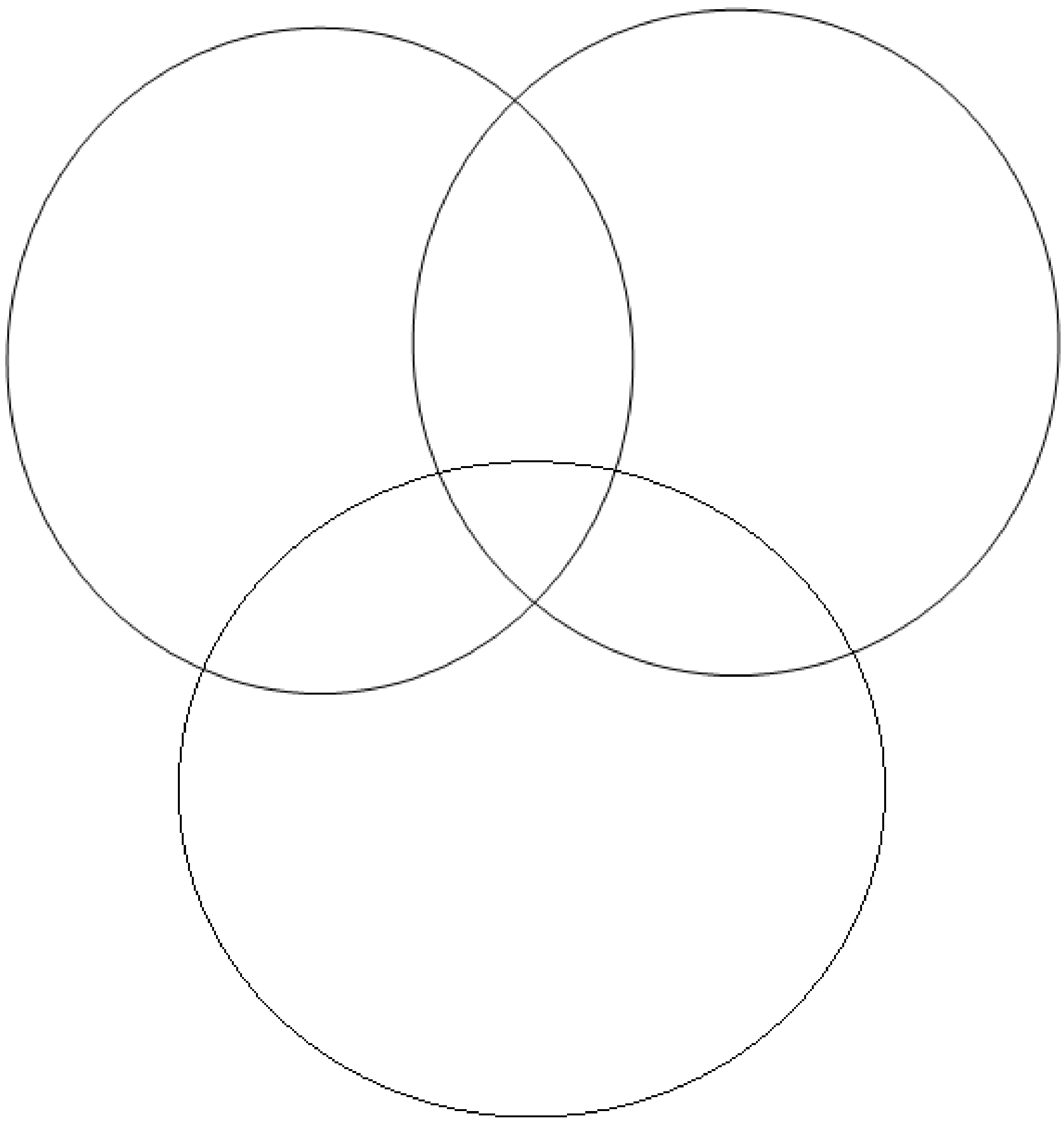
Appendix B

Sequence Chart



Appendix C

Venn Diagram



Appendix D

Meta-Cognitive Chart for Data Analysis

Metacognitive Chart for Analyzing Data

Directions:
As you complete each phase of step1 in the problem solving process, check the appropriate box below.

1. Read/Reread	☐
2. Paraphrase (restate in your own words)	☐
3. Visualize (Draw a picture to represent the problem)	☐
4. Worked with a group to discuss the problem.	☐

After you have check off all four steps, answer the following questions:

1. What is the problem asking or what question am I trying to answer?
2. What information is still missing?
3. What type of mathematical computation will I need to use to solve the problem?
4. Does everyone in my group agree on these answers?

If everyone does not agree, be sure to write down all extra responses in the space below (you may need them for step 2).

Appendix E

Estimation Workspace

Estimation Workspace for Problem Solving

Directions:

1. Write your hypothesis for the problem (you can write two if you have not decided on a final hypothesis).
2. In the workspace below your hypothesis, write your estimated answer. Check to see if it makes sense in the problem, and adjust it up or down.

Hypothesis #1	
Possible Solution #1	Possible Solution #2

Hypothesis #1	
Possible Solution #1	Possible Solution #2

Compare your answers with your group members and write down your final estimate below. ↓

Final Estimate

Appendix F

Implementing Solution Worksheet

Implementing Your Solution

Step 1. Solve the problem using your plan.

Step 2. Be sure to double check each step. 

Step 3. If the plan is not working after a few attempts, try a different plan. 

Step 4. Allow for mistakes (remember the plan may need some revision). 

Step 5. Check your answer. (Plug the answer back into the problem and decide if it makes sense). 

Appendix G

Solution Reflection Card

Reflect on your participation in the problem solving process:

My answer made sense because…

My method worked because…

I learned that I…

I was surprised that I…

Resources

Draw a Picture

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Montague, M. (2005). Math problem-solving for upper elementary students with disabilities. Retrieved from the Web February 29, 2008. http://www.k8accesscenter.org/training_resources/documents/Math%20Prob%20Solving%20Upper%20Elementary.pdf

Visualization

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52,* 215-241.

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Van Garderen, D. (2006). Spatial visualization, visual imagery, and mathematical problem-solving of students with varying abilities. *Journal of Learning Disabilities, 39*(6), 496-506.

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